

DEFORMATION THEORY OF K-THEORETIC CYCLES

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ABSTRACT. By using higher K-theory, we study deformation theory of K-theoretic cycles. As an application, we answer affirmatively the following two questions posed by Mark Green and Phillip Griffiths in Chap 10 of [11]:

- (1). How to define tangent space to algebraic cycles $TZ^q(X)$ in general ?
- (2). Obstruction issues(V.S. Hilbert scheme).

The highlight is the appearance of negative K-groups which detects the obstructions for deformation of cycles.

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1. Introduction

Let $f : \mathcal{X} \rightarrow S$ be a smooth projective morphism, where $S = \text{Spec}(k[[t]])$ and k is a field of characteristic 0. Let $X_j = \mathcal{X} \times_S S_j$,

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where $S_j = \text{Spec}(k[t]/t^{j+1})$, we have the following natural diagram

$$\begin{array}{ccccc} X_j & \xrightarrow{f_j} & X_{j+1} & \xrightarrow{i_{j+1}} & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ S_j & \longrightarrow & S_{j+1} & \longrightarrow & S. \end{array}$$

We use X to denote X_0 , and call the family $\{X_j\}_j$ a deformation of X , where X_j is called the j -th infinitesimal neighborhood of X . In particular, the family $\{X_j\}_j$ is a trivial deformation of X , if, for each j , $X_j = X \times_k S_j$.

The following question is posed by Green-Griffiths in [11]:

Question 1.1. *How to define tangent space to $Z^q(X)$ in general ?*

Since the abelian group $Z^q(X)$ is not a complex manifold or a scheme, so the known deformation theory, such as Kodaira-Spencer theory or the theory of Hilbert schemes, can't apply to this question directly. In our approach, $Z^q(-)$ is considered as a functor and we firstly attempt to define the tangent space to this functor as usual

$$TZ^q(X) := \text{Ker}\{Z^q(X \times \text{Spec}(k[\varepsilon])) \xrightarrow{\varepsilon=0} Z^q(X)\}.$$

Unfortunately, the classical definition of algebraic cycles can't distinguish nilpotent, $Z^q(X \times \text{Spec}(k[\varepsilon])) = Z^q(X)$, so this definition is clearly not the desired one.

Green-Griffiths has answered this question for $q = 1$ and $q = \dim(X)$ in [11] and points out that (page 186):

The technical issue that arises in trying straightforwardly extend the definitions given in the text for $p = n, 1$ concerns cycles that are linear combinations of irreducible subvarieties

$$Z = \sum_i n_i Z_i,$$

where some Z_i may not be the support of a locally Cohen-Maculay scheme.

To handle this technical issue, we look at generic points of Z_i s. Z_i s are generically Cohen-Maculay, though they may not be locally Cohen-Maculay. In order to realize this idea precisely, we need to use the formal language of higher K-theory. In Chapter 3, we answer this question affirmatively which generalizes Green-Griffiths' definitions, see Definition 3.2.

Considering an element $\tau \in TZ^q(X)$ as a first order deformation, Green-Griffiths asks whether we can successively deform τ to infinite

order. It is well-known that the deformation of a subvariety Y , considered as an element of the Hilbert scheme $\text{Hilb}(X)$, can be obstructed. However, Green-Griffiths predicts that we can eliminate obstructions by considering Y as an element of $Z^q(X)$:

Question 1.2. *$TZ^q(X)$ is unobstructed. See chapter 4 for precise statement.*

We answer this question affirmatively in chapter 4.

In [3], Balmer defines K-theoretic Chow groups in terms of the derived category $D^{\text{perf}}(X)$ obtained from the exact category of perfect complexes of \mathcal{O}_X -modules. His idea is followed by Klein [13] and the author [21]. We further push-forward the idea in [21] and answer Green-Griffiths' above questions affirmatively.

This note is organized as follows. We recall K-theoretic background and Milnor K-theoretic cycles in Section 2.1 and introduce deformation theory of K-theoretic cycles in Section 2.2. The relation between obstruction and negative K-groups is discussed in Section 2.3.

We answer Green-Griffiths' **Question 1.1** in Section 3.1 and compare our approach with Green-Griffiths' [11] in Section 3.2. In Section 3.3 and Section 3.4, we explain two new aspects of our Milnor K-theoretic cycles, featuring negative K-group and Milnor K-theory, which is different from Balmer's [3].

Green-Griffiths' **Question 1.2** is answered in chapter 4.

Notations and conventions.

(1). K-theory used in this note will be Thomason-Trobaugh non-connective K-theory [19], if not stated otherwise. For any abelian group M , $M_{\mathbb{Q}}$ denotes the image of M in $M \otimes_{\mathbb{Z}} \mathbb{Q}$. $\text{Spec}(k[\varepsilon])$ denotes the dual number, $\varepsilon^2 = 0$. We always assume X has finite Krull dimension d .

2. Deformation of cycles in general

2.1. Milnor K-theoretic Chow groups. Let X be a noetherian scheme of finite type over a field k of finite Krull dimension d . As explained in [2], one can filter the tensor triangulated category $\mathcal{L} = D^{\text{perf}}(X)$ by dimension of support

$$\cdots \subset \mathcal{L}_{(p)}(X) \subset \mathcal{L}_{(p+1)}(X) \subset \cdots \subset \mathcal{L},$$

where $\mathcal{L}_{(p)}(X)$ is defined to be

$$\mathcal{L}_{(p)}(X) := \{E \in D^{\text{perf}}(X) \mid \text{codim}_{\text{Krull}}(\text{supp}(E)) \geq -p\}.$$

Let $(\mathcal{L}_{(p)}(X)/\mathcal{L}_{(p-1)}(X))^{\#}$ denote the idempotent completion of the Verdier quotient $\mathcal{L}_{(p)}(X)/\mathcal{L}_{(p-1)}(X)$.

Theorem 2.1. [1] *For each $p \in \mathbb{Z}$, localization induces an equivalence*

$$(\mathcal{L}_{(p)}(X)/\mathcal{L}_{(p-1)}(X))^{\#} \simeq \bigsqcup_{x \in X^{(-p)}} D_x^{\text{perf}}(X)$$

between the idempotent completion of the quotient $\mathcal{L}_{(p)}(X)/\mathcal{L}_{(p-1)}(X)$ and the coproduct over $x \in X^{(-p)}$ of the derived category of perfect complexes of $O_{X,x}$ -modules with homology supported on the closed point $x \in \text{Spec}(O_{X,x})$.

The short sequence

$$\mathcal{L}_{(p-1)}(X) \rightarrow \mathcal{L}_{(p)}(X) \rightarrow (\mathcal{L}_{(p)}(X)/\mathcal{L}_{(p-1)}(X))^{\#},$$

which is exact up to summand, induces the following homotopy fibration of K-theory spectrum:

$$\mathcal{K}(\mathcal{L}_{(p-1)}(X)) \rightarrow \mathcal{K}(\mathcal{L}_{(p)}(X)) \rightarrow \mathcal{K}((\mathcal{L}_{(p)}(X)/\mathcal{L}_{(p-1)}(X))^{\#}).$$

As pointed in [2], this fibration gives rise to a long exact sequence:

$$\cdots \rightarrow K_n(\mathcal{L}_{(p-1)}(X)) \rightarrow K_n(\mathcal{L}_{(p)}(X)) \xrightarrow{i} K_n((\mathcal{L}_{(p)}(X)/\mathcal{L}_{(p-1)}(X))^{\#}) \xrightarrow{k} K_{n-1}(\mathcal{L}_{(p-1)}(X)) \rightarrow \cdots,$$

which produces an exact couple as usual and then gives rise to the associated coniveau spectral sequence with E_1 -term:

$$E_1^{p,q} = K_{-p-q}((\mathcal{L}_{(-p)}(X)/\mathcal{L}_{(-p-1)}(X))^{\#}),$$

the differential $d_1^{p,q}$ is the composition $d_1^{p,q} = i \circ k$ as usual

$$\begin{aligned} d_1^{p,q} : K_{-p-q}((\mathcal{L}_{(-p)}(X)/\mathcal{L}_{(-p-1)}(X))^{\#}) &\xrightarrow{k} K_{-p-q-1}(\mathcal{L}_{(-p-1)}(X)) \\ &\xrightarrow{i} K_{-p-q-1}((\mathcal{L}_{(-p-1)}(X)/\mathcal{L}_{(-p-2)}(X))^{\#}). \end{aligned}$$

Definition 2.2. [2] *For X a noetherian scheme of finite type over a field k of finite Krull dimension d and for each integer q satisfying $1 \leq q \leq d+1$, the q -th (augmented) Gersten complex, is defined to be the $(-q)$ -th line of E_1 -page of the above coniveau spectral sequence:*

$$\begin{aligned} G_q : 0 \rightarrow K_q(X) &\rightarrow \bigoplus_{x \in X^{(0)}} K_q(O_{X,x}) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(q-1)}} K_1(O_{X,x} \text{ on } x) \\ &\xrightarrow{d_{1,X}^{q-1,-q}} \bigoplus_{x \in X^{(q)}} K_0(O_{X,x} \text{ on } x) \xrightarrow{d_{1,X}^{q,-q}} \bigoplus_{x \in X^{(q+1)}} K_{-1}(O_{X,x} \text{ on } x) \rightarrow \cdots \\ &\rightarrow \bigoplus_{x \in X^{(d)}} K_{q-d}(O_{X,x} \text{ on } x) \rightarrow 0. \end{aligned}$$

Remark 2.3. *Since Thomason-Trobaugh K-spectrum satisfies Zariski excision, one can use the machinery of [7] to construct a coniveau spectral sequence and Consequently a Gersten complex. This has been used in [8, 20]. We thank anonymous comments on this.*

Keeping Green-Griffiths' Questions 1.1 and Questions 1.2 in mind, we feel that Balmer's construction is flexible for our purpose. Balmer's Theorem 2.1 guides us to look at generic points of irreducible subvarieties which is the key to answer Green-Griffiths' Questions 1.1. Moreover, taking idempotent complete in Theorem 2.1 results in the appearance of negative K-groups, which detects the obstructions for deformations of cycles, see Section 2.3.

Let $X[t, t^{-1}] = X \times_k k[t, t^{-1}]$, we have also the following homotopy fibration of K-theory spectrum:

$$\mathcal{K}(\mathcal{L}_{(p-1)}(X[t, t^{-1}])) \rightarrow \mathcal{K}(\mathcal{L}_{(p)}(X[t, t^{-1}])) \rightarrow \mathcal{K}((\mathcal{L}_{(p)}(X[t, t^{-1}])/\mathcal{L}_{(p-1)}(X[t, t^{-1}]))^{\#}).$$

Mimicking the argument in [19], chap 6, Lemma 6.3, we have the maps

$$\cup t : \mathcal{K}(\mathcal{L}_{(p-1)}(X)) \rightarrow \Omega \mathcal{K}(\mathcal{L}_{(p-1)}(X[t, t^{-1}])),$$

$$\cup t : \mathcal{K}(\mathcal{L}_{(p)}(X)) \rightarrow \Omega \mathcal{K}(\mathcal{L}_{(p)}(X[t, t^{-1}])).$$

These two maps induce the following one

$$\cup t : \mathcal{K}((\mathcal{L}_{(p)}(X)/\mathcal{L}_{(p-1)}(X))^{\#}) \rightarrow \Omega \mathcal{K}((\mathcal{L}_{(p)}(X[t, t^{-1}])/\mathcal{L}_{(p-1)}(X[t, t^{-1}]))^{\#}).$$

Consequently, we have the following map between coniveau spectral sequences

$$\cup t : E_1^{p,q}(X) \rightarrow E_1^{p,q-1}(X[t, t^{-1}]),$$

where $E_1^{p,q}(X) = K_{-p-q}((\mathcal{L}_{(-p)}(X)/\mathcal{L}_{(-p-1)}(X))^{\#})$ and $E_1^{p,q-1}(X[t, t^{-1}]) = K_{-p-q+1}((\mathcal{L}_{(-p)}(X[t, t^{-1}])/\mathcal{L}_{(-p-1)}(X[t, t^{-1}]))^{\#})$.

Therefore, $\cup t$ gives rise to the following commutative diagram between augmented Gersten complexes, from now on, we write Y to replace $X[t, t^{-1}]$ for simplicity,

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K_{q+1}(Y) & \xleftarrow{\cup t} & K_q(X) \\
\downarrow & & \downarrow \\
\bigoplus_{y \in Y^{(0)}} K_{q+1}(O_{Y,y}) & \xleftarrow{\vdots} & \bigoplus_{x \in X^{(0)}} K_{q+1}(O_{X,x}) \\
\downarrow & & \downarrow \\
\bigoplus_{y \in Y^{(1)}} K_q(O_{Y,y} \text{ on } y) & \xleftarrow{\quad} & \bigoplus_{x \in X^{(1)}} K_{q-1}(O_{X,x} \text{ on } x) \\
\downarrow & & \downarrow \\
\cdots & \xleftarrow{\quad} & \cdots \\
\downarrow & & \downarrow \\
\bigoplus_{y \in Y^{(q)}} K_1(O_{Y,y} \text{ on } y) & \xleftarrow{\quad} & \bigoplus_{x \in X^{(q)}} K_0(O_{X,x} \text{ on } x) \\
\downarrow d_{1,Y}^{q,-q-1} & & \downarrow d_{1,X}^{q,-q} \\
\bigoplus_{y \in Y^{(q+1)}} K_0(O_{Y,y} \text{ on } y) & \xleftarrow{\quad} & \bigoplus_{x \in X^{(q+1)}} K_{-1}(O_{X,x} \text{ on } x) \\
\downarrow & & \downarrow \\
\cdots & \xleftarrow{\quad} & \cdots \\
\downarrow & & \downarrow \\
\bigoplus_{y \in Y^{(d)}} K_{q-d+1}(O_{Y,y} \text{ on } y) & \xleftarrow{\quad} & \bigoplus_{x \in X^{(d)}} K_{q-d}(O_{X,x} \text{ on } x) \\
\downarrow & & \downarrow \\
\bigoplus_{y \in Y^{(d+1)}} K_{q-d}(O_{Y,y} \text{ on } y) & \xleftarrow{\quad} & 0 \\
\downarrow & & \\
0. & &
\end{array}$$

From now on, we assume X has an ample line bundle (so does Y), then the non-negative index part of the columns in the above diagram agrees with the Gersten type complexes associated to the coniveau

spectral sequences constructed by Gillet [9], page 239,

$$0 \rightarrow K_{q+1}(Y) \rightarrow \bigoplus_{y \in Y^{(0)}} K_{q+1}(O_{Y,y}) \rightarrow \cdots \rightarrow \bigoplus_{y \in Y^{(q)}} K_1(O_{Y,y} \text{ on } y) \rightarrow \bigoplus_{y \in Y^{(q+1)}} K_0(O_{Y,y} \text{ on } y) \rightarrow 0,$$

$$0 \rightarrow K_q(X) \rightarrow \bigoplus_{x \in X^{(0)}} K_q(O_{X,x}) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(q)}} K_0(O_{X,x} \text{ on } x) \rightarrow 0.$$

Adams operations ψ^k exist on $K_*(O_{X,x} \text{ on } x)$ and can decompose the above complexes into direct sum of subcomplexes respectively, according to the eigenweight,

$$0 \rightarrow K_{q+1}^{(j)}(Y) \rightarrow \bigoplus_{y \in Y^{(0)}} K_{q+1}^{(j)}(O_{Y,y}) \rightarrow \cdots \rightarrow \bigoplus_{y \in Y^{(q)}} K_1^{(j)}(O_{Y,y} \text{ on } y) \rightarrow \bigoplus_{y \in Y^{(q+1)}} K_0^{(j)}(O_{Y,y} \text{ on } y) \rightarrow 0,$$

$$0 \rightarrow K_q^{(j)}(X) \rightarrow \bigoplus_{x \in X^{(0)}} K_q^{(j)}(O_{X,x}) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(q)}} K_0^{(j)}(O_{X,x} \text{ on } x) \rightarrow 0.$$

Moreover, we have the following commutative diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ K_{q+1}^{(i+1)}(Y) & \xleftarrow{\cup t} & K_q^{(i)}(X) \\ \downarrow & & \downarrow \\ \bigoplus_{y \in Y^{(0)}} K_{q+1}^{(i+1)}(O_{Y,y}) & \xleftarrow{\vdots} & \bigoplus_{x \in X^{(0)}} K_q^{(i)}(O_{X,x}) \\ \downarrow & & \downarrow \\ \bigoplus_{y \in Y^{(1)}} K_q^{(i+1)}(O_{Y,y} \text{ on } y) & \xleftarrow{\quad} & \bigoplus_{x \in X^{(1)}} K_{q-1}^{(i)}(O_{X,x} \text{ on } x) \\ \downarrow & & \downarrow \\ \cdots & \xleftarrow{\quad} & \cdots \\ \downarrow & & \downarrow \\ \bigoplus_{y \in Y^{(q)}} K_1^{(i+1)}(O_{Y,y} \text{ on } y) & \xleftarrow{\quad} & \bigoplus_{x \in X^{(q)}} K_0^{(i)}(O_{X,x} \text{ on } x) \\ \downarrow d_{1,Y}^{q, -q-1} & & \downarrow d_{1,X}^{q, -q} \\ \bigoplus_{y \in Y^{(q+1)}} K_0^{(i+1)}(O_{Y,y} \text{ on } y) & \xleftarrow{\quad} & 0 \\ \downarrow & & \\ 0. & & \end{array}$$

It is known(due to Weibel) that one can extend Adams operations ψ^k to negative K-groups(with support) by using Bass fundamental exact sequence. So Adams operations ψ^k exists on each term of the augmented Gersten complex in Definition 2.2.

Proposition 2.4. *For X an equidimensional noetherian scheme of finite type over a field and having an ample line bundle, the differentials $d_1^{p,-q}$ of the augmented Gersten complex(Definition 2.2) respect Adams operations ψ^k . In other words, for every $i \in \mathbb{Z}$, there exists the following complex*

$$\begin{aligned} G_q^{(i)} : 0 \rightarrow K_q^{(i)}(X) &\rightarrow \bigoplus_{x \in X^{(0)}} K_q^{(i)}(O_{X,x}) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(q-1)}} K_1^{(i)}(O_{X,x} \text{ on } x) \\ &\xrightarrow{d_{1,X}^{q,-1,-q}} \bigoplus_{x \in X^{(q)}} K_0^{(i)}(O_{X,x} \text{ on } x) \xrightarrow{d_{1,X}^{q,-q}} \bigoplus_{x \in X^{(q+1)}} K_{-1}^{(i)}(O_{X,x} \text{ on } x) \rightarrow \cdots \\ &\rightarrow \bigoplus_{x \in X^{(d)}} K_{q-d}^{(i)}(O_{X,x} \text{ on } x) \rightarrow 0. \end{aligned}$$

Proof. It suffices to prove that, for each $p \geq q$, $d_{1,X}^{p,-q}$ respects Adams operations ψ^k . We check this for $d_{1,X}^{q,-q}$. Let $\eta \in \bigoplus_{x \in X^{(q)}} K_0^{(i)}(O_{X,x} \text{ on } x)$, then $\eta \cdot t \in \bigoplus_{y \in Y^{(q)}} K_1^{(i+1)}(O_{Y,y} \text{ on } y)$. Since $d_{1,Y}^{q,-q-1}$ respects Adams operations ψ^k , $d_{1,Y}^{q,-q-1}(\eta \cdot t) \in \bigoplus_{y \in Y^{(q+1)}} K_0^{(i+1)}(O_{Y,y} \text{ on } y)$. This means $d_{1,X}^{q,-q}(\eta) \cdot t (= d_{1,Y}^{q,-q-1}(\eta \cdot t)) \in \bigoplus_{y \in Y^{(q+1)}} K_0^{(i+1)}(O_{Y,y} \text{ on } y)$, so we have $d_{1,X}^{q,-q}(\eta) \in \bigoplus_{x \in X^{(q+1)}} K_{-1}^{(i)}(O_{X,x} \text{ on } x)$. \square

In particular, for $i = q$, we obtain the following complex

$$\begin{aligned} G_q^{(q)} : 0 \rightarrow K_q^{(q)}(X) &\rightarrow \bigoplus_{x \in X^{(0)}} K_q^{(q)}(O_{X,x}) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(q-1)}} K_1^{(q)}(O_{X,x} \text{ on } x) \\ &\xrightarrow{d_{1,X}^{q,-1,-q}} \bigoplus_{x \in X^{(q)}} K_0^{(q)}(O_{X,x} \text{ on } x) \xrightarrow{d_{1,X}^{q,-q}} \bigoplus_{x \in X^{(q+1)}} K_{-1}^{(q)}(O_{X,x} \text{ on } x) \rightarrow \cdots \\ &\rightarrow \bigoplus_{x \in X^{(d)}} K_{q-d}^{(q)}(O_{X,x} \text{ on } x) \rightarrow 0. \end{aligned}$$

Recall that Milnor K-groups with support are rationally defined in terms of eigenspaces of Adams operations in [21].

Definition 2.5. [21] *Let X be an equidimensional noetherian scheme of finite type over a field and having an ample line bundle. Let $x \in X^{(j)}$, Milnor K-group with support $K_m^M(O_{X,x} \text{ on } x)$ is rationally defined to be*

$$K_m^M(O_{X,x} \text{ on } x) := K_m^{(m+j)}(O_{X,x} \text{ on } x)_{\mathbb{Q}},$$

where $K_m^{(m+j)}$ is the eigenspace for $\psi^k = k^{m+j}$ and ψ^k is the Adams operations.

Similarly, Milnor K-group $K_m^M(X)$ is defined to be

$$K_m^M(X) := K_m^{(m)}(X)\mathbb{Q}.$$

Tensoring each term in the complex $G_q^{(q)}$ with \mathbb{Q} , we obtain the following complex

$$\begin{aligned} G_q^{(q)} : 0 \rightarrow K_q^M(X) \rightarrow \bigoplus_{x \in X^{(0)}} K_q^M(O_{X,x}) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(q-1)}} K_1^M(O_{X,x} \text{ on } x) \\ \xrightarrow{d_{1,X}^{q-1,-q}} \bigoplus_{x \in X^{(q)}} K_0^M(O_{X,x} \text{ on } x) \xrightarrow{d_{1,X}^{q,-q}} \bigoplus_{x \in X^{(q+1)}} K_{-1}^M(O_{X,x} \text{ on } x) \rightarrow \cdots \\ \rightarrow \bigoplus_{x \in X^{(d)}} K_{q-d}^M(O_{X,x} \text{ on } x) \rightarrow 0. \end{aligned}$$

This means the assumption in Definition 3.4 in [21] is satisfied, so we have:

Definition 2.6. [21] *For X an equidimensional noetherian scheme of finite type over a field and having an ample line bundle, the q -th Milnor K-theoretic cycles and Milnor K-theoretic rational equivalence, denoted $Z_q^M(D^{\text{Perf}}(X))$ and $Z_{q,\text{rat}}^M(D^{\text{Perf}}(X))$, are defined via the above complex $G_q^{(q)}$:*

$$\begin{aligned} Z_q^M(D^{\text{Perf}}(X)) &= \text{Ker}(d_{1,X}^{q,-q}), \\ Z_{q,\text{rat}}^M(D^{\text{Perf}}(X)) &= \text{Im}(d_{1,X}^{q-1,-q}). \end{aligned}$$

The q -th Milnor K-theoretic Chow group is defined to be:

$$CH_q^M(D^{\text{Perf}}(X)) = \frac{\text{Ker}(d_{1,X}^{q,-q})}{\text{Im}(d_{1,X}^{q-1,-q})}.$$

The reason why we take the kernel of $d_{1,X}^{q,-q}$ to define $Z_q^M(D^{\text{Perf}}(X))$ and why we use Milnor K-groups with support, i.e., certain eigenspaces of Thomason-Trobaugh K-groups, not entire Thomason-Trobaugh K-groups, are explained in Section 3.3 and Section 3.4 respectively.

2.2. Deformation of cycles. From now on, we turn to the situation introduced in the beginning of Section 1 and keep the notations f, f_j, i_{j+1} . The natural map

$$f_j : X_j \rightarrow X_{j+1},$$

induces $f_j^* : \mathcal{K}(X_{j+1}) \rightarrow \mathcal{K}(X_j)$. Moreover, f_j^* induces maps between coniveau spectral sequences recalled in Section 2.1:

$$f_j^* : E_1^{p,q}(X_{j+1}) \rightarrow E_1^{p,q}(X_j).$$

This gives us the following commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K_q(X_{j+1}) & \xrightarrow{f_j^*} & K_q(X_j) \\
\downarrow & & \downarrow \\
K_q(k(X)_{j+1}) & \xrightarrow{\vdots} & K_q(k(X)_j) \\
\downarrow & & \downarrow \\
\bigoplus_{x_{j+1} \in X_{j+1}^{(1)}} K_{q-1}(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \longrightarrow & \bigoplus_{x_j \in X_j^{(1)}} K_{q-1}(O_{X_j, x_j} \text{ on } x_j) \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow \\
\bigoplus_{x_{j+1} \in X_{j+1}^{(q-1)}} K_1(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \longrightarrow & \bigoplus_{x_j \in X_j^{(q-1)}} K_1(O_{X_j, x_j} \text{ on } x_j) \\
\downarrow d_{1, X_{j+1}}^{q-1, -q} & & \downarrow d_{1, X_j}^{q-1, -q} \\
\bigoplus_{x_{j+1} \in X_{j+1}^{(q)}} K_0(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \longrightarrow & \bigoplus_{x_j \in X_j^{(q)}} K_0(O_{X_j, x_j} \text{ on } x_j) \\
\downarrow d_{1, X_{j+1}}^{q, -q} & & \downarrow d_{1, X_j}^{q, -q} \\
\bigoplus_{x_{j+1} \in X_{j+1}^{(q+1)}} K_{-1}(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \longrightarrow & \bigoplus_{x_j \in X_j^{(q+1)}} K_{-1}(O_{X_j, x_j} \text{ on } x_j) \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow \\
\bigoplus_{x_{j+1} \in X_{j+1}^{(d)}} K_{q-d}(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \longrightarrow & \bigoplus_{x_j \in X_j^{(d)}} K_{q-d}(O_{X_j, x_j} \text{ on } x_j) \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}$$

Since Adams operations are natural with respect to pull-back, see Corollary 5.4 in [14], we have the following commutative diagram

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 K_q^{(i)}(X_{j+1}) & \xrightarrow{f_j^*} & K_q^{(i)}(X_j) \\
 \downarrow & & \downarrow \\
 K_q^{(i)}(k(X_{j+1})) & \xrightarrow{\quad \vdots \quad} & K_q^{(i)}(k(X_j)) \\
 \downarrow & & \downarrow \\
 \bigoplus_{x_{j+1} \in X_{j+1}^{(1)}} K_{q-1}^{(i)}(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \longrightarrow & \bigoplus_{x_j \in X_j^{(1)}} K_{q-1}^{(i)}(O_{X_j, x_j} \text{ on } x_j) \\
 \downarrow & & \downarrow \\
 \dots & \longrightarrow & \dots \\
 \downarrow & & \downarrow \\
 \bigoplus_{x_{j+1} \in X_{j+1}^{(q-1)}} K_1^{(i)}(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \longrightarrow & \bigoplus_{x_j \in X_j^{(q-1)}} K_1^{(i)}(O_{X_j, x_j} \text{ on } x_j) \\
 \downarrow d_{1, X_{j+1}}^{q-1, -q} & & \downarrow d_{1, X_j}^{q-1, -q} \\
 \bigoplus_{x_{j+1} \in X_{j+1}^{(q)}} K_0^{(i)}(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \longrightarrow & \bigoplus_{x_j \in X_j^{(q)}} K_0^{(i)}(O_{X_j, x_j} \text{ on } x_j) \\
 \downarrow d_{1, X_{j+1}}^{q, -q} & & \downarrow d_{1, X_j}^{q, -q} \\
 \bigoplus_{x_{j+1} \in X_{j+1}^{(q+1)}} K_{-1}^{(i)}(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \longrightarrow & \bigoplus_{x_j \in X_j^{(q+1)}} K_{-1}^{(i)}(O_{X_j, x_j} \text{ on } x_j) \\
 \downarrow & & \downarrow \\
 \dots & \longrightarrow & \dots \\
 \downarrow & & \downarrow \\
 \bigoplus_{x_{j+1} \in X_{j+1}^{(d)}} K_{q-d}^{(i)}(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \longrightarrow & \bigoplus_{x_j \in X_j^{(d)}} K_{q-d}^{(i)}(O_{X_j, x_j} \text{ on } x_j) \\
 \downarrow & & \downarrow \\
 0 & & 0.
 \end{array}$$

In particular, we have the following commutative diagram:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K_q^M(X_{j+1}) & \xrightarrow{f_j^*} & K_q^M(X_j) \\
\downarrow & & \downarrow \\
K_q^M(k(X_{j+1})) & \xrightarrow{\vdots} & K_q^M(k(X_j)) \\
\downarrow & & \downarrow \\
\bigoplus_{x_{j+1} \in X_{j+1}^{(1)}} K_{q-1}^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \longrightarrow & \bigoplus_{x_j \in X_j^{(1)}} K_{q-1}^M(O_{X_j, x_j} \text{ on } x_j) \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow \\
\bigoplus_{x_{j+1} \in X_{j+1}^{(q-1)}} K_1^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \longrightarrow & \bigoplus_{x_j \in X_j^{(q-1)}} K_1^M(O_{X_j, x_j} \text{ on } x_j) \\
\downarrow d_{1, X_{j+1}}^{q-1, -q} & & \downarrow d_{1, X_j}^{q-1, -q} \\
\bigoplus_{x_{j+1} \in X_{j+1}^{(q)}} K_0^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \longrightarrow & \bigoplus_{x_j \in X_j^{(q)}} K_0^M(O_{X_j, x_j} \text{ on } x_j) \\
\downarrow d_{1, X_{j+1}}^{q, -q} & & \downarrow d_{1, X_j}^{q, -q} \\
\bigoplus_{x_{j+1} \in X_{j+1}^{(q+1)}} K_{-1}^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \longrightarrow & \bigoplus_{x_j \in X_j^{(q+1)}} K_{-1}^M(O_{X_j, x_j} \text{ on } x_j) \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow \\
\bigoplus_{x_{j+1} \in X_{j+1}^{(d)}} K_{q-d}^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \longrightarrow & \bigoplus_{x_j \in X_j^{(d)}} K_{q-d}^M(O_{X_j, x_j} \text{ on } x_j) \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}$$

For each i and $x \in X^{(i)}$, we want to compute the kernel of

$$K_*(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) \xrightarrow{f_j^*} K_*(O_{X_j, x_j} \text{ on } x_j).$$

Infinitesimal lifting property says that X_j is **locally** a trivial deformation of X , see Lemma 2.4 in [6] for details. By choosing an open affine

neighborhood $x \in U \subset X$,

$$X_j|_U \cong U \times_k S_j = U[t]/(t^{j+1}),$$

where $S_j = \text{Spec}(k[t]/(t^{j+1}))$. Let $U_j = U[t]/(t^{j+1})$, we have

$$O_{X_j, x_j} \cong O_{U_j, x_j} \cong O_{U, x}[t]/(t^{j+1}) \cong O_{X, x}[t]/(t^{j+1}).$$

Similarly,

$$O_{X_{j+1}, x_{j+1}} \cong O_{U_{j+1}, x_{j+1}} \cong O_{U, x}[t]/(t^{j+2}) \cong O_{X, x}[t]/(t^{j+2}).$$

These isomorphisms say that

$$K_*(O_{X_j, x_j} \text{ on } x_j) \cong K_*(O_{X, x}[t]/(t^{j+1}) \text{ on } x[t]/(t^{j+1})),$$

and consequently

$$K_*^M(O_{X_j, x_j} \text{ on } x_j) \cong K_*^M(O_{X, x}[t]/(t^{j+1}) \text{ on } x[t]/(t^{j+1})),$$

For any integer m , let $K_m^M(O_{X_j, x_j} \text{ on } x_j, t)$ denote the relative K-group, that is, the kernel of the natural projection

$$K_m^M(O_{X_j, x_j} \text{ on } x_j) \xrightarrow{t=0} K_m^M(O_{X, x} \text{ on } x).$$

Then we have

$$K_m^M(O_{X_j, x_j} \text{ on } x_j, t) \cong K_m^M(O_{X, x}[t]/(t^{j+1}) \text{ on } x[t]/(t^{j+1}), t).$$

Recall that we have proved the following isomorphisms in [21]:

Theorem 2.7. [21] *Let X be a smooth projective variety over a field k of characteristic 0 and let $x \in X^{(i)}$. Chern character induces the following isomorphisms between relative K-groups and local cohomology groups:*

$$K_m^M(O_{X, x}[t]/(t^{j+1}) \text{ on } x[t]/(t^{j+1}), t) \cong H_x^i((\Omega_{O_{X, x}/\mathbb{Q}}^{m+i-1})^{\oplus j}).$$

Moreover, it is known that from the computation of Hochschild(cyclic) homology of truncated polynomials, $H_x^i((\Omega_{O_{X, x}/\mathbb{Q}}^{m+i-1})^{\oplus j})$ carries additional structure.

$$H_x^i((\Omega_{O_{X, x}/\mathbb{Q}}^{m+i-1})^{\oplus j}) \cong tH_x^i((\Omega_{O_{X, x}/\mathbb{Q}}^{m+i-1})) \oplus \cdots \oplus t^j H_x^i((\Omega_{O_{X, x}/\mathbb{Q}}^{m+i-1})).$$

To simplify the notations, we use A to denote $K_m^M(O_{X, x} \text{ on } x)$ and B to denote $H_x^i(\Omega_{O_{X, x}/\mathbb{Q}}^{m+i-1})$, then we have

$$K_m^M(O_{X, x}[t]/(t^{j+1}) \text{ on } x[t]/(t^{j+1})) \cong A \oplus tB \oplus \cdots \oplus t^j B.$$

There exists the following commutative diagram

$$\begin{array}{ccc}
K_m^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \xrightarrow{f_j^*} & K_m^M(O_{X_j, x_j} \text{ on } x_j) \\
\cong \downarrow & & \cong \downarrow \\
K_m^M(O_{X, x}[t]/(t^{j+2}) \text{ on } x[t]/(t^{j+2})) & \xrightarrow{f_j^*} & K_m^M(O_{X, x}[t]/(t^{j+1}) \text{ on } x[t]/(t^{j+1})) \\
\cong \downarrow & & \cong \downarrow \\
A \oplus tB \oplus \cdots t^j B \oplus t^{j+1} B & \xrightarrow{t^{j+1}=0} & A \oplus tB \oplus \cdots t^j B,
\end{array}$$

and the following short exact sequence of abelian groups:

$$0 \rightarrow B \rightarrow A \oplus tB \oplus \cdots t^j B \oplus t^{j+1} B \xrightarrow{t^{j+1}=0} A \oplus tB \oplus \cdots t^j B \rightarrow 0.$$

This shows that

Lemma 2.8. *For each integer i and $x \in X^{(i)}$, there exists the following short exact sequence of abelian groups, where m is any integer,*

$$0 \rightarrow H_x^i(\Omega_{O_{X, x}/\mathbb{Q}}^{m+i-1}) \rightarrow K_m^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) \xrightarrow{f_j^*} K_m^M(O_{X_j, x_j} \text{ on } x_j) \rightarrow 0.$$

Remark 2.9. *In private discussions, Spencer Bloch points out to the author that the isomorphism*

$$K_m^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) \cong K_m^M(O_{X, x}[t]/(t^{j+2}) \text{ on } x[t]/(t^{j+2}))$$

*is **not** canonical, it depends on the choice of open affine neighborhood U . This non-canonical isomorphism may have a crucial role in studying obstruction issues.*

The author acknowledges many helpful comments and questions from Spencer Bloch.

2.3. Obstructions and negative K-groups. For each j , the j -th infinitesimal neighborhood X_j satisfies the assumptions of Definition 2.6, so Definition 2.6 applies:

Corollary 2.10. *For the j -th infinitesimal neighborhood X_j , the q -th Milnor K -theoretic cycles and rational equivalence of X_j are defined to be*

$$Z_q^M(D^{\text{perf}}(X_j)) = \text{Ker}(d_{1, X_j}^{q, -q}), Z_{q, \text{rat}}^M(D^{\text{perf}}(X_j)) = \text{Im}(d_{1, X_j}^{q-1, -q}).$$

Definition 2.11. *Given $\xi_j \in Z_q^M(D^{\text{perf}}(X_j))$, an element $\xi_{j+1} \in Z_q^M(D^{\text{perf}}(X_{j+1}))$ is called a deformation of ξ_j , if $f_j^*(\xi_{j+1}) = \xi_j$.*

ξ_j and ξ_{j+1} can be formally written as finite sums

$$\sum_{x_j} \lambda_j \cdot \overline{\{x_j\}}_{\text{red}} \text{ and } \sum_{x_{j+1}} \lambda_{j+1} \cdot \overline{\{x_{j+1}\}}_{\text{red}},$$

where $\sum_{x_j} \lambda_j \in \text{Ker}(d_{1,X_j}^{q,-q}) \subset \bigoplus_{x_j \in X_j^{(q)}} K_0^M(O_{X_j, x_j} \text{ on } x_j)$ and $\overline{\{x_j\}}_{\text{red}}$ is the closed reduced scheme associated to $\overline{\{x_j\}}$.

Since $\overline{\{x_j\}}_{\text{red}} = \overline{\{x_{j+1}\}}_{\text{red}}$, when we deform from ξ_j to ξ_{j+1} , we really deform the **coefficients**, i.e, we deform from $\sum_{x_j} \lambda_j$ to $\sum_{x_{j+1}} \lambda_{j+1}$.

Since

$$f_j^* : \bigoplus_{x_{j+1} \in X_{j+1}^{(q)}} K_0^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) \rightarrow \bigoplus_{x_j \in X_j^{(q)}} K_0^M(O_{X_j, x_j} \text{ on } x_j)$$

is surjective, see lemma 2.8, given any $\xi_j \in Z_q^M(D^{\text{perf}}(X_j))$, there exists

$$\xi_{j+1} \in \bigoplus_{x_{j+1} \in X_{j+1}^{(q)}} K_0^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1})$$

such that $f_j^*(\xi_{j+1}) = \xi_j$. We would like to know whether $\xi_{j+1} \in Z_q^M(D^{\text{perf}}(X_{j+1}))$.

An easy diagram chasing shows $f_j^* d_{1,X_{j+1}}^{q,-q}(\xi_{j+1}) = 0$, so $d_{1,X_{j+1}}^{q,-q}(\xi_{j+1}) \in \text{Ker}(f_j^*) = \bigoplus_{x \in X^{(q+1)}} H_x^{q+1}(\Omega_{X/\mathbb{Q}}^{q-1})$. If $d_{1,X_{j+1}}^{q,-q}(\xi_{j+1}) = 0$, then we can lift ξ_j to ξ_{j+1} .

Definition 2.12. *The obstruction space for lifting elements in $Z_q^M(D^{\text{perf}}(X_j))$ to $Z_q^M(D^{\text{perf}}(X_{j+1}))$ is defined to be $\bigoplus_{x \in X^{(q+1)}} H_x^{q+1}(\Omega_{X/\mathbb{Q}}^{q-1})$.*

Definition 2.13. *Given $\eta_j \in Z_{q,\text{rat}}^M(D^{\text{perf}}(X_j))$, an element $\eta_{j+1} \in Z_{q,\text{rat}}^M(D^{\text{perf}}(X_{j+1}))$ is called a deformation of η_j , if $f_j^*(\eta_{j+1}) = \eta_j$.*

3. First order trivial deformation-tangent spaces

In this section, X is a smooth projective variety over a field k of characteristic 0. We look at trivial deformations of X and use it to study tangent spaces to $Z^q(X)$. To fix notations, $X_j = X \times_k S_j$ globally (this is always true locally). To emphasize on first order trivial deformation, we use $X[\varepsilon]$ to stand for X_1 , i.e., $X[\varepsilon] = (X, O_X[t]/(t^2))$.

3.1. Definition of tangent spaces. Recall that the tangent space to a functor \mathcal{F} , denoted $T\mathcal{F}(X)$, is defined to be

$$T\mathcal{F}(X) := \text{Ker}\{\mathcal{F}(X[\varepsilon]) \xrightarrow{\varepsilon=0} \mathcal{F}(X)\}.$$

We recall that the Milnor K-theoretic cycles and Chow groups in Definition 2.6 recover the classical ones for X :

Theorem 3.1. *Compatiblity* [21]

$$Z_q^M(D^{\text{perf}}(X)) = Z^q(X)_{\mathbb{Q}},$$

$$Z_{q,\text{rat}}^M(D^{\text{perf}}(X)) = Z_{\text{rat}}^q(X)_{\mathbb{Q}},$$

$$CH_q^M(D^{\text{perf}}(X)) = CH^q(X)_{\mathbb{Q}}.$$

This guides us to the following definition, which answers Green-Griffiths' **Question 1.1** affirmatively:

Definition 3.2. *The tangent space to q -cycles, denoted $TZ^q(X)$, is defined to be*

$$TZ^q(X) := TZ_q^M(D^{\text{perf}}(X)) = \text{Ker}\{Z_q^M(D^{\text{perf}}(X[\varepsilon])) \xrightarrow{\varepsilon=0} Z_q^M(D^{\text{perf}}(X))\}.$$

Similarly, the tangent space to rational equivalent classes, denoted $TZ_{\text{rat}}^q(X)$, is defined to be

$$TZ_{\text{rat}}^q(X) := TZ_{q,\text{rat}}^M(D^{\text{perf}}(X)) = \text{Ker}\{Z_{q,\text{rat}}^M(D^{\text{perf}}(X[\varepsilon])) \xrightarrow{\varepsilon=0} Z_{q,\text{rat}}^M(D^{\text{perf}}(X))\}.$$

The following theorem has been proved in [21]:

Theorem 3.3. [21] *There exists the following commutative diagram in which the Zariski sheafification of each column is a flasque resolution of $\Omega_{X/\mathbb{Q}}^{q-1}$, $K_q^M(O_{X[\varepsilon]})$ and $K_q^M(O_X)$ respectively. The left arrows are induced by Chern characters from K -theory to negative cyclic homology*

and the right ones are the natural maps sending ε to 0:

$$\begin{array}{ccccc}
 0 & & 0 & & 0 \\
 \downarrow & & \downarrow & & \downarrow \\
 \Omega_{k(X)/\mathbb{Q}}^{q-1} & \xleftarrow{\text{Chern}} & K_q^M(k(X)[\varepsilon]) & \xrightarrow[\varepsilon=0]{f_1^*} & K_q^M(k(X)) \\
 \downarrow & & \downarrow & & \downarrow \\
 \bigoplus_{x \in X^{(1)}} H_x^1(\Omega_{X/\mathbb{Q}}^{q-1}) & \xleftarrow{\vdots} & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(1)}} K_{q-1}^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) & \xrightarrow{\vdots} & \bigoplus_{x \in X^{(1)}} K_{q-1}^M(O_{X,x} \text{ on } x) \\
 \downarrow & & \downarrow & & \downarrow \\
 \dots & \xleftarrow{\quad} & \dots & \xrightarrow{\quad} & \dots \\
 \downarrow & & \downarrow & & \downarrow \\
 \bigoplus_{x \in X^{(q-1)}} H_x^{q-1}(\Omega_{X/\mathbb{Q}}^{q-1}) & \xleftarrow{\quad} & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(q-1)}} K_1^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) & \xrightarrow{\quad} & \bigoplus_{x \in X^{(q-1)}} K_1^M(O_{X,x} \text{ on } x) \\
 \partial_1^{q-1, -q} \downarrow & & d_{1, X_1}^{q-1, -q} \downarrow & & d_{1, X}^{q-1, -q} \downarrow \\
 \bigoplus_{x \in X^{(q)}} H_x^q(\Omega_{X/\mathbb{Q}}^{q-1}) & \xleftarrow{\quad} & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(q)}} K_0^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) & \xrightarrow{\quad} & \bigoplus_{x \in X^{(q)}} K_0^M(O_{X,x} \text{ on } x) \\
 \partial_1^{q, -q} \downarrow & & d_{1, X_1}^{q, -q} \downarrow & & d_{1, X}^{q, -q} \downarrow \\
 \bigoplus_{x \in X^{(q+1)}} H_x^{q+1}(\Omega_{X/\mathbb{Q}}^{q-1}) & \xleftarrow{\quad} & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(q+1)}} K_{-1}^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) & \xrightarrow{\quad} & \bigoplus_{x \in X^{(q+1)}} K_{-1}^M(O_{X,x} \text{ on } x) \\
 \downarrow & & \downarrow & & \downarrow \\
 \dots & \xleftarrow{\quad} & \dots & \xrightarrow{\quad} & \dots \\
 \downarrow & & \downarrow & & \downarrow \\
 \bigoplus_{x \in X^{(d)}} H_x^d(\Omega_{X/\mathbb{Q}}^{q-1}) & \xleftarrow{\quad} & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(d)}} K_{q-d}^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) & \xrightarrow{\quad} & \bigoplus_{x \in X^{(d)}} K_{q-d}^M(O_{X,x} \text{ on } x) \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0.
 \end{array}$$

This diagram enables us to compute $TZ^q(X)$ and $TZ_{rat}^q(X)$. A quick diagram chasing shows

Theorem 3.4.

$$TZ^q(X) = \text{Ker}(\partial_1^{q, -q}),$$

$$TZ_{rat}^q(X) = \text{Im}(\partial_1^{q-1, -q}).$$

Evidently, $TZ_{rat}^q(X)$ is a subspace of $TZ^q(X)$. We use the quotient space to define the tangent space to Chow groups:

Definition 3.5. *The tangent space to $CH^q(X)$, denoted $TCH^q(X)$, is defined to be*

$$TCH^q(X) := \frac{TZ^q(X)}{TZ_{rat}^q(X)}.$$

Theorem 3.6. *$TCH^q(X)$ agrees with the formal tangent space $T_fCH^q(X)$ defined by Bloch, where $T_fCH^q(X) = H^q(X, \Omega_{X/\mathbb{Q}}^{q-1})$.*

Proof. It immediately follows from the fact that the Zariski sheafification of the left column in Theorem 3.3 is a flasque resolution of $\Omega_{X/\mathbb{Q}}^{q-1}$. \square

3.2. Comparison with Green-Griffiths' work. We compare with Green-Griffiths' work [11]. To fix notations, X is a smooth projective surface over a field k of characteristic 0. In [11], Green and Griffiths define the tangent space $TZ^2(X)$ to the 0-cycles on X and a tangent subspace $TZ_{rat}^2(X)$ to the rational equivalence class by concrete geometric consideration.

To be precise, Green and Griffiths consider the following commutative diagram :

$$\begin{array}{ccccc}
 0 & & 0 & & 0 \\
 \downarrow & & \downarrow & & \downarrow \\
 \Omega_{k(X)/\mathbb{Q}}^1 & \longleftarrow & K_2^M(k(X)[\varepsilon]) & \xrightarrow{\varepsilon=0} & K_2^M(k(X)) \\
 \downarrow & & \downarrow & & \downarrow \\
 \bigoplus_{y \in X^{(1)}} H_y^1(\Omega_{X/\mathbb{Q}}^1) & \xleftarrow[\text{map}]{\text{tangent}} & \text{Arcs}^1(X) & \xrightarrow{\varepsilon=0} & \bigoplus_{y \in X^{(1)}} K_1^M(k(y)) \\
 \downarrow & & \downarrow & & \downarrow \\
 \bigoplus_{x \in X^{(2)}} H_x^2(\Omega_{X/\mathbb{Q}}^1) & \xleftarrow[\text{map}]{\text{tangent}} & \text{Arcs}^2(X) & \xrightarrow{\varepsilon=0} & \bigoplus_{x \in X^{(2)}} K_0^M(k(x)) \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0,
 \end{array}$$

where $\text{Arcs}^1(X)$ and $\text{Arcs}^2(X)$ stand for deformations of $\bigoplus_{y \in X^{(1)}} K_1^M(k(y))$

and $\bigoplus_{x \in X^{(2)}} K_0^M(k(x))$ respectively.

Green and Griffiths implicitly introduce groups of $\text{Arcs}^1(X)$ and $\text{Arcs}^2(X)$.

Definition 3.7. [11] *An element of $\text{Arcs}^2(X)$ is of the form $\text{Spec}(O_{X,x}[\varepsilon]/(u + \varepsilon u_1, v + \varepsilon v_1))$, where $\text{Spec}(O_{X,x}/(u, v))$ is supported on x and $u_1, v_1 \in O_{X,x}$.*

Definition 3.8. [11] *The tangent map: $\text{Arcs}^2(X) \rightarrow \bigoplus_{x \in X^{(2)}} H_x^2(\Omega_{X/\mathbb{Q}}^1)$ is defined in the following way. First, to $\text{Spec}(O_{X,x}[\varepsilon]/(u + \varepsilon u_1, v + \varepsilon v_1))$, the following diagram*

$$(3.1) \quad \begin{cases} O_{X,x} & \xrightarrow{(v, -u)^T} & O_{X,x}^{\oplus 2} & \xrightarrow{(u, v)} & O_{X,x} & \longrightarrow & O_{X,x}/(u, v) & \longrightarrow & 0 \\ O_{X,x} & \xrightarrow{v_1 du - u_1 dv} & \Omega_{O_{X,x}/\mathbb{Q}}^1 & & & & & & \end{cases}$$

gives an element β in $\text{Ext}_{O_{X,x}}^2(O_{X,x}/(u, v), \Omega_{O_{X,x}/\mathbb{Q}}^1)$. Noting that

$$H_x^2(\Omega_{X/\mathbb{Q}}^1) = \varinjlim_{n \rightarrow \infty} \text{Ext}_{O_{X,x}}^2(O_{X,x}/(u, v)^n, \Omega_{O_{X,x}/\mathbb{Q}}^1),$$

the image $[\beta]$ of β under the limit is in $H_x^2(\Omega_{X/\mathbb{Q}}^1)$ and it is defined to be the image of $\text{Spec}(O_{X,x}[\varepsilon]/(u + \varepsilon u_1, v + \varepsilon v_1))$ under the tangent map.

Suppose Y is an irreducible curve on X with generic point y . Let f be the local defining equations for Y and take a point $x \in Y$, then the local ring of y can be identified with the localization of the local ring of x with respect to (f) :

$$(3.2) \quad O_{X,y} = (O_{X,x})_{(f)}.$$

The following definition is used by Green-Griffiths [11], see chap 8, page 127.

Definition 3.9. [11] *Let $\text{div}(f + \varepsilon f_1)$ denote $\text{Spec}((O_{X,x})_{(f)}[\varepsilon]/(f + \varepsilon f_1))$, where $f_1 \in O_{X,x}$. An element of $\text{Arcs}^1(X)$ should be a formal sum of pairs of the form*

$$\{\text{div}(f + \varepsilon f_1), g + \varepsilon g_1 \mid \text{div}(f + \varepsilon f_1)\},$$

where $g + \varepsilon g_1 \in (K(Y)[\varepsilon])^$ and furthermore we assume $\text{div}(f + \varepsilon f_1)$ and $\text{div}(g + \varepsilon g_1)$ have no common curve components.*

Definition 3.10. [11] *The tangent map: $\text{Arcs}^1(X) \rightarrow \bigoplus_{y \in X^{(1)}} H_y^1(\Omega_{X/\mathbb{Q}}^1)$ is defined in the following way. For the pair $\{\text{div}(f + \varepsilon f_1), g + \varepsilon g_1 \mid \text{div}(f + \varepsilon f_1)\}$, the following diagram*

$$(3.3) \quad \begin{cases} (O_{X,x})_{(f)} & \xrightarrow{f} & (O_{X,x})_{(f)} & \longrightarrow & (O_{X,x})_{(f)}/(f) & \longrightarrow & 0 \\ (O_{X,x})_{(f)} & \xrightarrow{\frac{g_1 df}{g} - \frac{f_1 dg}{g}} & \Omega_{(O_{X,x})_{(f)}/\mathbb{Q}}^1 & & & & \end{cases}$$

gives an element α in $Ext_{(O_{X,x})_{(f)}}^1((O_{X,x})_{(f)}/(f), \Omega_{(O_{X,x})_{(f)}/\mathbb{Q}}^1)$. Noting that

$$H_y^1(\Omega_{X/\mathbb{Q}}^1) = \varinjlim_{n \rightarrow \infty} Ext_{(O_{X,x})_{(f)}}^1((O_{X,x})_{(f)}/(f)^n, \Omega_{(O_{X,x})_{(f)}/\mathbb{Q}}^1),$$

the image $[\alpha]$ of α under the limit is in $H_y^1(\Omega_{X/\mathbb{Q}}^1)$ and it is defined to be the image of $\{\text{div}(f + \varepsilon f_1), g + \varepsilon g_1 \mid \text{div}(f + \varepsilon f_1)\}$ under the tangent map.

With the above preparation, Green and Griffiths define the tangent space $TZ^2(X)$ to the 0-cycles on X and a tangent subspace $TZ_{rat}^2(X)$ to the rational equivalence class as follows:

Definition 3.11. [11] page 84, 141

$$TZ^2(X) = \bigoplus_{x \in X^{(2)}} H_x^2(\Omega_{X/\mathbb{Q}}^1), TZ_{rat}^2(X) = \text{Im}(\partial_1^{1,-2}),$$

where $\partial_1^{1,-2}$ is the differential of the Cousin complex in Theorem 3.12 below.

In order to compare with Green-Griffiths' approach, we need the following diagram:

Theorem 3.12. For $q = 2$ in Theorem 3.3, we have the following commutative diagram:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega_{k(X)/\mathbb{Q}}^1 & \longleftarrow & K_2^M(k(X)[\varepsilon]) & \xrightarrow{\varepsilon=0} & K_2^M(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{y \in X^{(1)}} H_y^1(\Omega_{X/\mathbb{Q}}^1) & \longleftarrow & \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} K_1^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) & \xrightarrow{\varepsilon=0} & \bigoplus_{y \in X^{(1)}} K_1^M(k(y)) \\
\partial_1^{1,-2} \downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x \in X^{(2)}} H_x^2(\Omega_{X/\mathbb{Q}}^1) & \longleftarrow & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(2)}} K_0^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) & \xrightarrow{\varepsilon=0} & \bigoplus_{x \in X^{(2)}} K_0^M(k(x)) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array}$$

Recall that Milnor K-group with support are defined as certain eigenspaces of K-groups, see Definition 2.5,

$$\begin{aligned}
K_1^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) &:= K_1^{(2)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}}; \\
K_0^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) &= K_0^{(2)}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon])_{\mathbb{Q}}.
\end{aligned}$$

Lemma 3.13.

$$K_1^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) = K_1(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}};$$

$$K_0^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) = K_0(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon])_{\mathbb{Q}}.$$

Proof. Since $K_1(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}} = K_1(O_{X,y} \text{ on } y)_{\mathbb{Q}} \oplus H_y^1(\Omega_{X/\mathbb{Q}}^1)$, one can check that $K_1^{(j)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}} = 0$, except for $j = 2$. That is,

$$K_1^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) = K_1(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}}.$$

Similar argument for $K_0^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon])$. \square

So the above diagram may be written as:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega_{k(X)/\mathbb{Q}}^1 & \longleftarrow & K_2^M(k(X)[\varepsilon]) & \xrightarrow{\varepsilon=0} & K_2^M(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{y \in X^{(1)}} H_y^1(\Omega_{X/\mathbb{Q}}^1) & \longleftarrow & \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} K_1(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) & \xrightarrow{\varepsilon=0} & \bigoplus_{y \in X^{(1)}} K_1^M(k(y)) \\
\partial_1^{1,-2} \downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x \in X^{(2)}} H_x^2(\Omega_{X/\mathbb{Q}}^1) & \longleftarrow & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(2)}} K_0(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) & \xrightarrow{\varepsilon=0} & \bigoplus_{x \in X^{(2)}} K_0^M(k(x)) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array}$$

In order to identify elements of $\text{Arcs}^1(X)$ and $\text{Arcs}^2(X)$ as elements of K-groups with support, we recall the following theorem, *Exercise 5.7* in [19]:

Theorem 3.14. [19] *Let X be a scheme with an ample family of line bundles. Let $i : Y \rightarrow X$ be a regular closed immersion ([SGA 6] VII Section 1) defined by ideal J . Suppose Y has codimension k in X . Then $K(X \text{ on } Y)$ is homotopy equivalent to the Quillen K-theory of the exact category of pseudo-coherent O_X -modules supported on the subspace Y and of Tor-dimension $\leq k$ on X .*

Back to our situation, X is a surface and x is a point on X . According to this theorem, $K_0(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon])$ can be considered as a K-group of the exact category of pseudo-coherent $O_{X,x}[\varepsilon]$ -modules supported on the subspace $x[\varepsilon]$ and of Tor-dimension ≤ 2 on $O_{X,x}[\varepsilon]$. Any element

of $\text{Arcs}^2(X)$ is such a module, and can be considered as an element of $K_0(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon])$. So we have:

Lemma 3.15.

$$\text{Arcs}^2(X) \subseteq \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(2)}} K_0(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]).$$

Similarly, $K_1(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])$ can be considered as a K-group of the exact category of pseudo-coherent $O_{X,y}[\varepsilon]$ -modules supported on the subspace $y[\varepsilon]$ and of Tor-dimension ≤ 1 on $O_{X,y}[\varepsilon]$. $(O_{X,x}(f)[\varepsilon]/(f + \varepsilon f_1))$ is such a module.

Considering $g + \varepsilon g_1$ as an automorphism of $(O_{X,x}(f)[\varepsilon]/(f + \varepsilon f_1))$, we can associate a double short exact sequence (in the sense of [15]) to $\{\text{div}(f + \varepsilon f_1), g + \varepsilon g_1 \mid \text{div}(f + \varepsilon f_1)\}$

$$(3.4) \quad \begin{cases} 0 \rightarrow (O_{X,x}(f)[\varepsilon]/(f + \varepsilon f_1)) \xrightarrow{1} (O_{X,x}(f)[\varepsilon]/(f + \varepsilon f_1)) \rightarrow 0 \\ 0 \rightarrow (O_{X,x}(f)[\varepsilon]/(f + \varepsilon f_1)) \xrightarrow{g + \varepsilon g_1} (O_{X,x}(f)[\varepsilon]/(f + \varepsilon f_1)) \rightarrow 0. \end{cases}$$

This double short exact sequence defines an element of $K_1(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])$, see [15] for details. So we have:

Lemma 3.16.

$$\text{Arcs}^1(X) \subseteq \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} K_1(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]).$$

By taking $q = 2$ in Theorem 3.4, we immediately see that Green and Griffiths' definitions agree with our Definition 3.2:

Corollary 3.17. *For X a smooth projective surface, Green and Griffiths' definitions $TZ^2(X)$ and $TZ_{rat}^2(X)$ agree with our Definition 3.2.*

3.3. Why take kernel. In this subsection, combining with Green-Griffiths' results in [11], we explain the geometric significance of taking kernel of $d_{1,X_1}^{q,-q}$ to define $Z_q^M(D^{\text{perf}}(X[\varepsilon]))$ in Definition 2.6, instead of taking $\bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(q)}} K_0^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon])$.

To fix notations, X is a smooth projective surface over a field k of characteristic 0.

Theorem 3.18. *For $q = 1$ in Theorem 3.3, we have the following commutative diagram:*

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
k(X) & \xleftarrow{\text{Chern}} & K_1^M(k(X)[\varepsilon]) & \xrightarrow{\varepsilon=0} & K_1^M(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{y \in X^{(1)}} H_y^1(O_X) & \xleftarrow{\text{Chern}} & \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) & \xrightarrow{\varepsilon=0} & \bigoplus_{y \in X^{(1)}} K_0^M(k(y)) \\
\partial_1^{1,-1} \downarrow & & d_{1,X_1}^{1,-1} \downarrow & & \downarrow \\
\bigoplus_{x \in X^{(2)}} H_x^2(O_X) & \xleftarrow[\cong]{\text{Chern}} & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(2)}} K_{-1}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) & \xrightarrow{\varepsilon=0} & \bigoplus_{x \in X^{(2)}} K_{-1}^M(k(x)) = 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array}$$

Remark 3.19. *The reason why we can use $K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])$ and $K_{-1}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon])$ to replace $K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])$ and $K_{-1}^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon])$ may be explained (in a similar way) by Lemma 3.13.*

In general, we don't know how to describe the elements of $K_{-1}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon])$, though we have shown $K_{-1}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon])$ is isomorphic to $H_x^2(O_X)$. This results in that it is very difficult to describe $\text{Ker}(d_{1,X_1}^{1,-1})$ explicitly.

Combining this commutative diagram with Green-Griffiths' results in [11], chap 8, page 102, we construct a concrete element in $\text{Ker}(d_{1,X_1}^{1,-1})$.

Let Y_1 and Y_2 be two curves on X with generic point y_1 and y_2 respectively. For simplicity, we work locally in Zariski topology and assume Y_1 and Y_2 intersect transversely at a point x . Around the point x , we can write

$$Y_1 = \text{div}(f_1); \quad Y_2 = \text{div}(f_2).$$

Take $g \in O_{X,x}$ such that $g(x) \neq 0$, we consider $O_{X,x}[\varepsilon]/(f_1 f_2 + \varepsilon g)$. The Koszul resolution of $O_{X,x}[\varepsilon]/(f_1 f_2 + \varepsilon g)$,

$$L^\bullet : 0 \rightarrow O_{X,x}[\varepsilon] \xrightarrow{f_1 f_2 + \varepsilon g} O_{X,x}[\varepsilon],$$

defines an element of $K_0(\mathcal{L}_{-1}(X[\varepsilon])/\mathcal{L}_{-2}(X[\varepsilon]))^\#$.

Theorem 3.20. $L^\bullet \in \text{Ker}(d_{1,X_1}^{1,-1})$.

Proof. Under the isomorphism in Theorem 2.1

$$K_0((\mathcal{L}_{(-1)}(X[\varepsilon])/\mathcal{L}_{(-2)}(X[\varepsilon]))^\#) \simeq \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} K_0(D_{y[\varepsilon]}^{\text{perf}}(X[\varepsilon])),$$

L^\bullet decomposes into the direct sum of

$$L_1^\bullet : 0 \rightarrow (O_{X,x})_{(f_1)}[\varepsilon] \xrightarrow{f_1 + \varepsilon \frac{g}{f_2}} (O_{X,x})_{(f_1)}[\varepsilon]$$

and

$$L_2^\bullet : 0 \rightarrow (O_{X,x})_{(f_2)}[\varepsilon] \xrightarrow{f_2 + \varepsilon \frac{g}{f_1}} (O_{X,x})_{(f_2)}[\varepsilon].$$

Noting $O_{X,y_1} = (O_{X,x})_{(f_1)}$, we have $L_1^\bullet \in K_0(O_{X,y_1}[\varepsilon] \text{ on } y_1[\varepsilon])$. Similarly, $L_2^\bullet \in K_0(O_{X,y_2}[\varepsilon] \text{ on } y_2[\varepsilon])$.

The following diagram, associated to L_1^\bullet ,

$$(3.5) \quad \begin{cases} (O_{X,x})_{(f_1)} \xrightarrow{f_1} (O_{X,x})_{(f_1)} \longrightarrow (O_{X,x})_{(f_1)}/(f_1) \longrightarrow 0 \\ (O_{X,x})_{(f_1)} \xrightarrow{\frac{g}{f_2}} (O_{X,x})_{(f_1)}, \end{cases}$$

gives an element α in $Ext_{O_{X,y_1}}^1(O_{X,y_1}/(f_1), O_{X,y_1})$. Noting that

$$H_{y_1}^1(O_X) = \varinjlim_{n \rightarrow \infty} Ext_{O_{X,y_1}}^1(O_{X,y_1}/(f_1)^n, O_{X,y_1}),$$

the image $[\alpha]$ of α under the limit is in $H_{y_1}^1(O_X)$ and it is the image of L_1^\bullet under the Chern map.

Similarly, the following diagram, associated to L_2^\bullet ,

$$(3.6) \quad \begin{cases} (O_{X,x})_{(f_2)} \xrightarrow{f_2} (O_{X,x})_{(f_2)} \longrightarrow (O_{X,x})_{(f_2)}/(f_2) \longrightarrow 0 \\ (O_{X,x})_{(f_2)} \xrightarrow{\frac{g}{f_1}} (O_{X,x})_{(f_2)}, \end{cases}$$

gives an element β in $Ext_{O_{X,y_2}}^1(O_{X,y_2}/(f_2), O_{X,y_2})$. Noting that

$$H_{y_2}^1(O_X) = \varinjlim_{n \rightarrow \infty} Ext_{O_{X,y_2}}^1(O_{X,y_2}/(f_2)^n, O_{X,y_2}),$$

the image $[\beta]$ of β under the limit is in $H_{y_2}^1(O_X)$ and it is the image of L_2^\bullet under the Chern map.

According to Green-Griffiths [11], page 103, $\partial_1^{1,-1}$ maps α in $H_x^2(O_X)$ to :

$$(3.7) \quad \begin{cases} O_{X,x} \xrightarrow{(f_2, -f_1)^T} O_{X,x}^{\oplus 2} \xrightarrow{(f_1, f_2)} O_{X,x} \longrightarrow O_{X,x}/(f_1, f_2) \longrightarrow 0 \\ O_{X,x} \xrightarrow{g} O_{X,x}. \end{cases}$$

Similarly, $\partial_1^{1,-1}$ maps β in $H_x^2(O_X)$ to :

$$(3.8) \quad \begin{cases} O_{X,x} \xrightarrow{(f_1, -f_2)^T} O_{X,x}^{\oplus 2} \xrightarrow{(f_2, f_1)} O_{X,x} \longrightarrow O_{X,x}/(f_1, f_2) \longrightarrow 0 \\ O_{X,x} \xrightarrow{g} O_{X,x}. \end{cases}$$

Noting the commutative diagram below

$$\begin{array}{ccccccc} O_{X,x} & \xrightarrow{(f_2, -f_1)^T} & O_{X,x}^{\oplus 2} & \xrightarrow{(f_1, f_2)} & O_{X,x} & \longrightarrow & O_{X,x}/(f_1, f_2) \longrightarrow 0 \\ -1 \downarrow & & M \downarrow & & 1 \downarrow & & = \downarrow \\ O_{X,x} & \xrightarrow{(f_1, -f_2)^T} & O_{X,x}^{\oplus 2} & \xrightarrow{(f_2, f_1)} & O_{X,x} & \longrightarrow & O_{X,x}/(f_2, f_1) \longrightarrow 0, \end{array}$$

where M stands for the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

Green-Griffiths observes that $\partial_1^{1,-1}(\alpha)$ and $\partial_1^{1,-1}(\beta)$ are negative of each other in $Ext_{O_{X,x}}^2(O_{X,x}/(f_1, f_2), O_{X,x})$. Hence, $\partial_1^{1,-2}(\alpha + \beta)$ is 0 in $H_x^2(O_X)$. Therefore, $d_{1,X_1}^{1,-1}(L^\bullet) = 0$ because of the commutative diagram:

$$\begin{array}{ccc} \bigoplus_{y \in X^{(1)}} H_y^1(O_X) & \xleftarrow{\text{Chern}} & \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \\ \partial_1^{1,-1} \downarrow & & d_{1,X_1}^{1,-1} \downarrow \\ \bigoplus_{x \in X^{(2)}} H_x^2(O_X) & \xleftarrow[\cong]{\text{Chern}} & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(2)}} K_{-1}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]). \end{array}$$

□

The above argument seems formal, so it's convenient to have an intuitive explanation the meaning of taking the kernel of $d_{1,X_1}^{1,-1}$. This has been done by using residue by Green-Griffiths [11].

Alternative explanation by using residue, due to Green-Griffiths [11](Section 8.2) To fix notations, let Y_1 and Y_2 be two curves on X . It is well-known that tangent vectors to the curves Y_1 and Y_2 are given by normal vector fields,

$$v_1 \in H^0(N_{Y_1/X}), v_2 \in H^0(N_{Y_2/X}).$$

For simplicity, we work locally in Zariski topology and assume Y_1 and Y_2 intersect transversely at a point x . Around the point x , we can write

$$Y_1 = \text{div}(f_1); \quad Y_2 = \text{div}(f_2).$$

Then v_1 and v_2 can be expressed as

$$v_1 = w_1 \frac{\partial}{\partial f_1}, v_2 = w_2 \frac{\partial}{\partial f_2},$$

for some functions w_1 and w_2 . For our purpose, we take $w_1 = \frac{g}{f_2}$ and

$w_2 = \frac{h}{f_1}$, then

$$v_1 = \frac{g}{f_2} \frac{\partial}{\partial f_1}, v_2 = \frac{h}{f_1} \frac{\partial}{\partial f_2}.$$

For $\omega = df_1 \wedge df_2$, we consider the Poincaré residue:

$$(3.9) \quad \begin{cases} v_1 \rfloor \omega = \text{Res}_{Y_1} \left(\frac{g df_1 \wedge df_2}{f_1 f_2} \right) = \frac{g df_2}{f_2} \in \Omega_{K(Y_1)/\mathbb{C}}^1; \\ v_2 \rfloor \omega = \text{Res}_{Y_2} \left(\frac{h df_1 \wedge df_2}{f_1 f_2} \right) = -\frac{h df_1}{f_1} \in \Omega_{K(Y_2)/\mathbb{C}}^1. \end{cases}$$

We further take the residue at x :

$$\text{Res}_x \left(\frac{g df_2}{f_2} \right) = g, \text{Res}_x \left(-\frac{h df_1}{f_1} \right) = -h.$$

The sum of the residues is

$$\text{Res}_x \left(\frac{g df_2}{f_2} \right) + \text{Res}_x \left(-\frac{h df_1}{f_1} \right) = g - h.$$

when $g = h$, the sum of the residues is 0:

Lemma 3.21. [11] For $v_1 = \frac{g}{f_2} \frac{\partial}{\partial f_1}$ and $v_2 = \frac{g}{f_1} \frac{\partial}{\partial f_2}$,

$$\text{Res}_x(v_1 \rfloor \omega) + \text{Res}_x(v_2 \rfloor \omega) = 0.$$

How does this connect to K-groups?

For normal vectors

$$v_1 = \frac{g}{f_2} \frac{\partial}{\partial f_1}, v_2 = \frac{g}{f_1} \frac{\partial}{\partial f_2},$$

v_1 corresponds to $f_1 + \varepsilon \frac{g}{f_2}$ and v_2 corresponds to $f_2 + \varepsilon \frac{g}{f_1}$. In other words, v_1 corresponds to the complex

$$L_1^\bullet : 0 \rightarrow (O_{X,x})_{(f_1)}[\varepsilon] \xrightarrow{f_1 + \varepsilon \frac{g}{f_2}} (O_{X,x})_{(f_1)}[\varepsilon]$$

and v_2 corresponds to the complex

$$L_2^\bullet : 0 \rightarrow (O_{X,x})_{(f_2)}[\varepsilon] \xrightarrow{f_2 + \varepsilon \frac{g}{f_1}} (O_{X,x})_{(f_2)}[\varepsilon].$$

Conclusion: $\text{Res}_x(v_1]\omega) + \text{Res}_x(v_2]\omega) = 0$ in Lemma 3.21 corresponds to $(L_1^\bullet + L_2^\bullet) \in \text{Ker}(d_{1,X_1}^{1,-1})$ in Theorem 3.20.

Remark 3.22. *One may ask why there is no necessary to take kernel in Quillen's or Soulé's proofs of Bloch's formula in [17, 18]. That's because negative K-groups are zero in this case, $K_{-1}(k(x)) = 0$. Even we take kernel, the cycles class group $Z^q(X)$ is still identified with $\bigoplus_{x \in X^{(q)}} K_0(k(x))$.*

3.4. Why use Milnor K-theory. In the following, we explain why we use Milnor K-groups with support, i.e., certain eigenspaces of Thomason-Trobaugh K-groups, not entire Thomason-Trobaugh K-groups, to define cycles and Chow groups in Definition 2.6.

In 2012 Fall, the author met a question on describing certain eigenspaces of K-groups and he E-mailed this question to Christophe Soulé for help. In the replying E-mail, Christophe Soulé suggested that if the author's question is true, then it should be true for Milnor K-theory and guided the author to read Theorem 5 in [18]:

In our setting, X is smooth projective over k , so the Gersten complex has the form of

$$\begin{aligned} 0 \rightarrow K_q(X) &\rightarrow \bigoplus_{x \in X^{(0)}} K_q(O_{X,x}) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(q-1)}} K_1(O_{X,x} \text{ on } x) \\ &\rightarrow \bigoplus_{x \in X^{(q)}} K_0(O_{X,x} \text{ on } x) \rightarrow 0, \end{aligned}$$

which agrees with the Gersten complex by Quillen [17] because of Dévissage:

$$\begin{aligned} 0 \rightarrow K_q(X) &\rightarrow \bigoplus_{x \in X^{(0)}} K_q(O_{X,x}) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(q-1)}} K_1(k(x)) \\ &\rightarrow \bigoplus_{x \in X^{(q)}} K_0(k(x)) \rightarrow 0. \end{aligned}$$

For $x \in X^{(q)}$, Adams operations can decompose $K_0(O_{X,x} \text{ on } x)$ and $K_0(k(x))$ into direct sums of eigenspaces respectively. Moreover, Riemann-Roch without denominator, due to Soulé [18], says

$$K_0^{(j)}(O_{X,x} \text{ on } x)_{\mathbb{Q}} = K_0^{(j-q)}(k(x))_{\mathbb{Q}}.$$

For $j = q$,

$$K_0^{(q)}(O_{X,x} \text{ on } x)_{\mathbb{Q}} = K_0^{(0)}(k(x))_{\mathbb{Q}} = K_0(k(x))_{\mathbb{Q}},$$

This forces to

$$K_0^{(j)}(O_{X,x} \text{ on } x)_{\mathbb{Q}} = 0, \text{ for } j \neq q.$$

So only $K_0^{(q)}(O_{X,x} \text{ on } x)_{\mathbb{Q}}$ is needed to study $Z^q(X)_{\mathbb{Q}}$.

The above idea become clear when we look at 0-cycles on a 3-fold.
To be precise,

Theorem 3.23. *Let X be a smooth projective variety over a field k of characteristic 0. For $q = 3$, $j = 1$, $i = 3$ and 2 in Theorem 3.13 of [21], we obtain the following two commutative diagrams:
The highest eigen-component*

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega_{k(X)/\mathbb{Q}}^2 & \longleftarrow & K_3^{(3)}(k(X)[\varepsilon]) & \xrightarrow{\varepsilon=0} & K_3^{(3)}(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{z \in X^{(1)}} H_z^1(\Omega_{X/\mathbb{Q}}^2) & \longleftarrow & \bigoplus_{z[\varepsilon] \in X[\varepsilon]^{(1)}} K_2^{(3)}(O_{X,z}[\varepsilon] \text{ on } z[\varepsilon]) & \xrightarrow{\varepsilon=0} & \bigoplus_{z \in X^{(1)}} K_2^{(3)}(O_{X,z} \text{ on } z) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{y \in X^{(2)}} H_y^2(\Omega_{X/\mathbb{Q}}^2) & \longleftarrow & \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(2)}} K_1^{(3)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) & \xrightarrow{\varepsilon=0} & \bigoplus_{y \in X^{(2)}} K_1^{(3)}(O_{X,y} \text{ on } y) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x \in X^{(3)}} H_x^3(\Omega_{X/\mathbb{Q}}^2) & \longleftarrow & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(3)}} K_0^{(3)}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) & \xrightarrow{\varepsilon=0} & \bigoplus_{x \in X^{(2)}} K_0^{(3)}(O_{X,x} \text{ on } x) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0,
\end{array}$$

and the second highest eigen-component

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
k(X) & \longleftarrow & K_3^{(2)}(k(X)[\varepsilon]) & \xrightarrow{\varepsilon=0} & K_3^{(2)}(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{z \in X^{(1)}} H_z^1(O_X) & \longleftarrow & \bigoplus_{z[\varepsilon] \in X[\varepsilon]^{(1)}} K_2^{(2)}(O_{X,z}[\varepsilon] \text{ on } z[\varepsilon]) & \xrightarrow{\varepsilon=0} & \bigoplus_{z \in X^{(1)}} K_2^{(2)}(O_{X,z} \text{ on } z) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{y \in X^{(2)}} H_y^2(O_X) & \longleftarrow & \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(2)}} K_1^{(2)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) & \xrightarrow{\varepsilon=0} & \bigoplus_{y \in X^{(2)}} K_1^{(2)}(O_{X,y} \text{ on } y) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x \in X^{(3)}} H_x^3(O_X) & \longleftarrow & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(3)}} K_0^{(2)}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) & \xrightarrow{\varepsilon=0} & \bigoplus_{x \in X^{(2)}} K_0^{(2)}(O_{X,x} \text{ on } x) = 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array}$$

We note $K_0^{(2)}(O_{X,x} \text{ on } x)_{\mathbb{Q}} = 0$ and

$$K_0^{(2)}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon])_{\mathbb{Q}} = H_x^3(O_X).$$

Because of depth condition, $H_x^3(O_X) \neq 0$, so $K_0^{(2)}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon])_{\mathbb{Q}}$ is non-zero. Therefore, in the second highest eigen-component diagram, the middle column can't be a deformation of the right column in the usual sense. In conclusion, only the highest eigen-component diagram is needed for studying deformation of cycles.

4. Obstruction issues-Versus Hilbert scheme

Let $Y \subset X$ be a subvariety of codimension q , we consider the trivial deformation $\{X_j\}_j$ of X , i.e., $X_j = X \times_k S_j$ for each j . Obstruction issues asks whether it is possible to lift Y to Y_j successively, where $Y_j \subset X_j$ with suitable assumptions.

It is a common phenomenon that obstructions can occur when doing deformation, though the deformation of X is trivial. For example, considering Y as an element of $\text{Hilb}(X)$, it is well known that the tangent space $T_Y \text{Hilb}(X)$ may be obstructed.

However, Green-Griffiths predicts that we can eliminate obstructions in their program [10, 11]. That is, considering Y as an element of $Z^q(X)$, $T_Y Z^q(X)$ is unobstructed.

Question 4.1. [11] page 187 – 190

There are essentially four (not mutually exclusive) possibilities:

- (i) $TZ^q(X)$ may be obstructed. That is, there exists some $\tau \in TZ^q(X)$ such that, thinking of τ as a map

$$\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2)) \rightarrow Z^q(X),$$

this map cannot be lifted to a map

$$\mathrm{Spec}(k[\varepsilon]/(\varepsilon^{k+1})) \rightarrow Z^q(X)$$

for some $k \geq 2$.

- (ii) $TZ^q(X)$ is unobstructed. That is, for any $\tau \in TZ^q(X)$, τ may be lifted to a map

$$\lim(\mathrm{Spec}(k[\varepsilon]/(\varepsilon^{k+1}))) \rightarrow Z^q(X).$$

- (iii) $TZ^q(X)$ is unobstructed, but there exists $\tau \in TZ^q(X)$ which is not the tangent to a geometric arc in $Z^q(X)$.

- (iv) Every $\tau \in TZ^q(X)$ is the tangent to a geometric arc in $Z^q(X)$.

The above question is expressed in a way, as if $Z^q(X)$ were a scheme. In fact, we know $Z^q(X)$ can't be treated as a scheme. In order to re-state this question, we introduce the following notations.

The natural map $i_j : X_j \rightarrow \mathcal{X}$ induces

$$i_j^* : Z_q^M(D^{\mathrm{Perf}}(\mathcal{X})) \rightarrow Z_q^M(D^{\mathrm{Perf}}(X_j)).$$

This gives rise to the following

$$(i_j^*) : Z_q^M(D^{\mathrm{Perf}}(\mathcal{X})) \rightarrow \varprojlim Z_q^M(D^{\mathrm{Perf}}(X_j)).$$

Definition 4.2. An element $\gamma \in \varprojlim Z_q^M(D^{\mathrm{Perf}}(X_j))$ is called a geometric arc if it lies in the image of (i_j^*) .

Green-Griffiths' question can be re-stated as follows:

Question 4.3. [11]

There are essentially four (not mutually exclusive) possibilities:

- (i) $TZ^q(X)$ may be obstructed. That is, there exists some $\tau \in TZ^q(X)$ such that τ cannot be lifted to $\tau_j \in Z_q^M(D^{\text{perf}}(X_j))$ for some j .
- (ii) $TZ^q(X)$ is unobstructed. That is, for any $\tau \in TZ^q(X)$, τ can be lifted to $\tau_j \in Z_q^M(D^{\text{perf}}(X_j))$ successively, where $j = 1, 2, \dots$. In other words, τ can be lifted to a $(\tau_j)_j \in \varprojlim Z_q^M(D^{\text{perf}}(X_j))$.
- (iii) $TZ^q(X)$ is unobstructed, but there exists $\tau \in TZ^q(X)$ which is not the tangent to a geometric arc in $Z^q(X)$. That is, the element $(\tau_j)_j \in \varprojlim Z_q^M(D^{\text{perf}}(X_j))$ in (ii) is not a geometric arc. See Definition 4.2 for the definition of geometric arc.
- (iv) Every $\tau \in TZ^q(X)$ is the tangent to a geometric arc in $Z^q(X)$. That is, τ can be lifted to $(\tau_j)_j \in \varprojlim Z_q^M(D^{\text{perf}}(X_j))$ and $(\tau_j)_j$ is a geometric arc.

For $q = 1$, this question was solved by TingFai Ng, a student of Phillip Griffiths and Mark Goresky, in his Ph.D thesis.

Theorem 4.4. [16]

Every $\tau \in TZ^1(X)$ is the tangent to a geometric arc in $Z^1(X)$.

Because of the known examples occurring in $q \geq 2$ in Chap 10 of [11](page 189) and in Section 4 of [10], Green-Griffiths observed that

Theorem 4.5. [11]

For $q \geq 2$, there exists X and $\tau \in TZ^q(X)$ which is not the tangent to a geometric arc in $Z^q(X)$.

This means only possibilities (i)-(iii) can occur for $q \geq 2$. Green-Griffiths conjectures that

Question 4.6. [11] page 190

(ii) and (iii) above are the only possibilities that actually occur for $q \geq 2$.

To answer this question, we need the following theorem:

Theorem 4.7. [21] *For $X_j = X \times_k \text{Spec}(k[t]/(t^{j+1}))$, where j is any integer, there exists the following commutative diagram:*

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
(\Omega_{k(X)/\mathbb{Q}}^{q-1})^{\oplus j} & \xleftarrow{\text{Chern}} & K_q^M(k(X_j)) & \xrightarrow{f_j^*} & K_q^M(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x \in X^{(1)}} H_x^1((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j}) & \xleftarrow{\text{Chern}} & \bigoplus_{x_j \in X_j^{(1)}} K_{q-1}^M(O_{X_j, x_j} \text{ on } x_j) & \xrightarrow{\vdots} & \bigoplus_{x \in X^{(1)}} K_{q-1}^M(O_{X, x} \text{ on } x) \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & \xleftarrow{\quad} & \vdots & \xrightarrow{\quad} & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x \in X^{(q-1)}} H_x^{q-1}((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j}) & \xleftarrow{\text{Chern}} & \bigoplus_{x_j \in X_j^{(q-1)}} K_1^M(O_{X_j, x_j} \text{ on } x_j) & \longrightarrow & \bigoplus_{x \in X^{(q-1)}} K_1^M(O_{X, x} \text{ on } x) \\
(\partial_1^{q-1, -q})^{\oplus j} \downarrow & & d_{1, X_j}^{q-1, -q} \downarrow & & d_{1, X}^{q-1, -q} \downarrow \\
\bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j}) & \xleftarrow{\text{Chern}} & \bigoplus_{x_j \in X_j^{(q)}} K_0^M(O_{X_j, x_j} \text{ on } x_j) & \longrightarrow & \bigoplus_{x \in X^{(q)}} K_0^M(O_{X, x} \text{ on } x) \\
(\partial_1^{q, -q})^{\oplus j} \downarrow & & d_{1, X_j}^{q, -q} \downarrow & & d_{1, X}^{q, -q} \downarrow \\
\bigoplus_{x \in X^{(q+1)}} H_x^{q+1}((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j}) & \xleftarrow[\cong]{\text{Chern}} & \bigoplus_{x_j \in X_j^{(q+1)}} K_{-1}^M(O_{X_j, x_j} \text{ on } x_j) & \longrightarrow & \bigoplus_{x \in X^{(q+1)}} K_{-1}^M(O_{X, x} \text{ on } x) = 0 \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & \xleftarrow{\quad} & \vdots & \xrightarrow{\quad} & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x \in X^{(d)}} H_x^d((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j}) & \xleftarrow{\text{Chern}} & \bigoplus_{x_j \in X_j^{(d)}} K_{q-d}^M(O_{X_j, x_j} \text{ on } x_j) & \longrightarrow & \bigoplus_{x \in X^{(d)}} K_{q-d}^M(O_{X, x} \text{ on } x) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array}$$

Using this theorem, we answer Green-Griffiths' **Question 1.2** affirmatively:

Theorem 4.8. $TZ^q(X)$ is unobstructed.

Proof. For any integer j and given any $\xi_j \in Z_q^M(D^{\text{perf}}(X_j)) (= \text{Ker}(d_{1, X_j}^{q, -q}))$, we need to show ξ_j can be lifted to an element of $Z_q^M(D^{\text{perf}}(X_{j+1})) (= \text{Ker}(d_{1, X_{j+1}}^{q, -q}))$. It is obvious that $\text{Chern}(\xi_j) \in \text{Ker}((\partial_1^{q, -q})^{\oplus j})$ because of

the commutative diagram (part of the diagram in Theorem 4.7),

$$\begin{array}{ccc}
\bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j}) & \xleftarrow{\text{Chern}} & \bigoplus_{x_j \in X_j^{(q)}} K_0^M(O_{X_j, x_j} \text{ on } x_j) \\
(\partial_1^{q, -q})^{\oplus j} \downarrow & & d_{1, X_j}^{q, -q} \downarrow \\
\bigoplus_{x \in X^{(q+1)}} H_x^{q+1}((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j}) & \xleftarrow[\cong]{\text{Chern}} & \bigoplus_{x_j \in X_j^{(q+1)}} K_{-1}^M(O_{X_j, x_j} \text{ on } x_j).
\end{array}$$

There exists a similar commutative diagram as in Theorem 4.7 for $j+1$. For our purpose, we need part of it:

$$\begin{array}{ccccc}
\bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j+1}) & \xleftarrow{\text{Chern}} & \bigoplus_{x_{j+1} \in X_{j+1}^{(q)}} K_0^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \longrightarrow & \bigoplus_{x \in X^{(q)}} K_0^M(O_{X, x} \text{ on } x) \\
(\partial_1^{q, -q})^{\oplus j+1} \downarrow & & d_{1, X_{j+1}}^{q, -q} \downarrow & & d_{1, X}^{q, -q} \downarrow \\
\bigoplus_{x \in X^{(q+1)}} H_x^{q+1}((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j+1}) & \xleftarrow[\cong]{\text{Chern}} & \bigoplus_{x_{j+1} \in X_{j+1}^{(q+1)}} K_{-1}^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \longrightarrow & \bigoplus_{x \in X^{(q+1)}} K_{-1}^M(O_{X, x} \text{ on } x) = 0
\end{array}$$

As explained on page 13-14, $\bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j})$ carries additional structure

$$\bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j}) \cong t \bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})) \oplus \cdots \oplus t^j \bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})).$$

And the differential

$$(\partial_1^{q, -q})^{\oplus j} : \bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j}) \rightarrow \bigoplus_{x \in X^{(q+1)}} H_x^{q+1}((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j})$$

splits as $t\partial_1^{q, -q} \oplus \cdots \oplus t^j\partial_1^{q, -q}$:

$$\begin{array}{ccc}
\bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j}) & \xrightarrow[\cong]{} & t \bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})) \oplus \cdots \oplus t^j \bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})) \\
(\partial_1^{q, -q})^{\oplus j} \downarrow & & t\partial_1^{q, -q} \oplus \cdots \oplus t^j\partial_1^{q, -q} \downarrow \\
\bigoplus_{x \in X^{(q+1)}} H_x^{q+1}((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j}) & \xrightarrow[\cong]{} & t \bigoplus_{x \in X^{(q+1)}} H_x^{q+1}((\Omega_{X/\mathbb{Q}}^{q-1})) \oplus \cdots \oplus t^j \bigoplus_{x \in X^{(q+1)}} H_x^{q+1}((\Omega_{X/\mathbb{Q}}^{q-1})),
\end{array}$$

where $\partial_1^{q, -q} : \bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})) \rightarrow \bigoplus_{x \in X^{(q+1)}} H_x^{q+1}((\Omega_{X/\mathbb{Q}}^{q-1}))$.

Under these isomorphisms, $\text{Chern}(\xi_j)$ can be written as $ta_1 + \cdots + t^j a_j$, where each $a_i \in \bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1}))$ and $\partial_1^{q, -q}(a_i) = 0$.

There exists a similar isomorphism for $j+1$:

$$\bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j+1}) \cong t \bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})) \oplus \cdots \oplus t^{j+1} \bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})).$$

And the differential

$$(\partial_1^{q,-q})^{\oplus j+1} : \bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j+1}) \rightarrow \bigoplus_{x \in X^{(q+1)}} H_x^{q+1}((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j+1})$$

splits as $t\partial_1^{q,-q} \oplus \dots \oplus t^{j+1}\partial_1^{q,-q}$.

We can always lift $ta_1 + \dots + t^j a_j$ to $ta_1 + \dots + t^j a_j + t^{j+1} a_{j+1}$ (note $t^{j+1} \neq 0$ here), where $a_{j+1} \in \text{Ker}(\partial_1^{q,-q})$. So $ta_1 + \dots + t^j a_j + t^{j+1} a_{j+1} \in \text{Ker}((\partial_1^{q,-q})^{\oplus j+1})$. Hence, we can always lift $\text{Chern}(\xi_j)$ to $\eta_{j+1} \in \text{Ker}((\partial_1^{q,-q})^{\oplus j+1})$.

Since the Chern map

$$\text{Chern} : \bigoplus_{x_{j+1} \in X_{j+1}^{(q)}} K_0^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) \rightarrow \bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j+1})$$

is surjective, there exists $\xi_{j+1} \in \bigoplus_{x_{j+1} \in X_{j+1}^{(q)}} K_0^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1})$ such

that $\text{Chern}(\xi_{j+1}) = \eta_{j+1}$.

By the naturality of Chern character, there exists the following commutative diagram:

$$\begin{array}{ccc} \bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j+1}) & \xleftarrow{\text{Chern}} & \bigoplus_{x_{j+1} \in X_{j+1}^{(q)}} K_0^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) \\ \downarrow t^{j+1}=0 & & \downarrow t^{j+1}=0 \\ \bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j}) & \xleftarrow{\text{Chern}} & \bigoplus_{x_j \in X_j^{(q)}} K_0^M(O_{X_j, x_j} \text{ on } x_j). \end{array}$$

So we have the following commutative diagram:

$$\begin{array}{ccc} \eta_{j+1} = \text{Chern}(\xi_{j+1}) & \xleftarrow{\text{Chern}} & \xi_{j+1} \\ \downarrow t^{j+1}=0 & & \downarrow t^{j+1}=0 \\ \eta_{j+1}|_{t^{j+1}=0} & \xleftarrow{\text{Chern}} & \xi_{j+1}|_{t^{j+1}=0}. \end{array}$$

This says $\eta_{j+1}|_{t^{j+1}=0} = \text{Chern}(\xi_{j+1}|_{t^{j+1}=0})$. On the other hand, since η_{j+1} lifts $\text{Chern}(\xi_j)$, $\eta_{j+1}|_{t^{j+1}=0} = \text{Chern}(\xi_j)$. Hence,

$$\xi_{j+1}|_{t^{j+1}=0} - \xi_j \in \text{Kernel of Chern} = \bigoplus_{x \in X^{(q)}} K_0^M(O_{X,x} \text{ on } x).$$

In other words, $\xi_{j+1}|_{t^{j+1}=0} = \xi_j + W$, for some $W \in \bigoplus_{x \in X^{(q)}} K_0^M(O_{X,x} \text{ on } x)$.

As a **cycle**, ξ_j can be written as a formal sum

$$(4.1) \quad \xi_j = (\xi_j + W) - W.$$

Since $\bigoplus_{x \in X^{(q)}} K_0^M(O_{X,x} \text{ on } x)$ is a direct summand of $\bigoplus_{x_{j+1} \in X_{j+1}^{(q)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1})$,
 $W \in \bigoplus_{x_{j+1} \in X_{j+1}^{(q)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1})$ and the cycle $\xi_{j+1} - W \in \bigoplus_{x_{j+1} \in X_{j+1}^{(q)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1})$ satisfies

$$(\xi_{j+1} - W)|_{t^{j+1}=0} = \xi_{j+1}|_{t^{j+1}=0} - W = \xi_j + W - W = \xi_j.$$

Moreover, $\text{Chern}(\xi_{j+1} - W) = \text{Chern}(\xi_{j+1}) = \eta_{j+1} \in \text{Ker}((\partial_1^{q,-q})^{\oplus j+1})$,
hence, $\xi_{j+1} - W \in Z_q^M(D^{\text{perf}}(X_{j+1})) (:= \text{Ker}(d_{1,X_{j+1}}^{q,-q}))$ because of the commutative diagram

$$\begin{array}{ccc} \bigoplus_{x \in X^{(q)}} H_x^q((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j+1}) & \xleftarrow{\text{Chern}} & \bigoplus_{x_{j+1} \in X_{j+1}^{(q)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \\ \downarrow (\partial_1^{q,-q})^{\oplus j+1} & & \downarrow d_{1,X_{j+1}}^{q,-q} \\ \bigoplus_{x \in X^{(q+1)}} H_x^{q+1}((\Omega_{X/\mathbb{Q}}^{q-1})^{\oplus j+1}) & \xleftarrow[\cong]{\text{Chern}} & \bigoplus_{x_{j+1} \in X_{j+1}^{(q+1)}} K_{-1}^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}). \end{array}$$

In conclusion, $\xi_{j+1} - W \in Z_q^M(D^{\text{perf}}(X_{j+1})) (:= \text{Ker}(d_{1,X_{j+1}}^{q,-q}))$ can lift ξ_j . \square

In Section 4 of [10], Green-Griffiths conjectures that

Question 4.9. [10] $TZ_{\text{rat}}^q(X)$ is unobstructed.

For any integer j and given any $\eta_j \in Z_{q,\text{rat}}^M(D^{\text{perf}}(X_j)) (:= \text{Im}(d_{1,X_j}^{q-1,-q}))$,
we want to know whether η_j can be lifted to $\eta_{j+1} \in Z_{q,\text{rat}}^M(D^{\text{perf}}(X_{j+1}))$.

By definition, $\eta_j = d_{1,X_j}^{q-1,-q}(\xi_j)$, for some $\xi_j \in \bigoplus_{x_j \in X_j^{(q-1)}} K_1^M(O_{X_j,x_j} \text{ on } x_j)$.

Since

$$f_j^* : \bigoplus_{x_{j+1} \in X_{j+1}^{(q-1)}} K_1^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \rightarrow \bigoplus_{x_j \in X_j^{(q-1)}} K_1^M(O_{X_j,x_j} \text{ on } x_j)$$

is surjective, see lemma 2.8, we can always lift ξ_j to $\xi_{j+1} \in \bigoplus_{x_{j+1} \in X_{j+1}^{(q-1)}} K_1^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1})$.

Then $d_{1, X_{j+1}}^{q-1, -q}(\xi_{j+1})$ lifts η_j because of the following commutative diagram:

$$\begin{array}{ccc} \bigoplus_{x_{j+1} \in X_{j+1}^{(q-1)}} K_1^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \xrightarrow{f_j^*} & \bigoplus_{x_j \in X_j^{(q-1)}} K_1^M(O_{X_j, x_j} \text{ on } x_j) \\ d_{1, X_{j+1}}^{q-1, -q} \downarrow & & d_{1, X_j}^{q-1, -q} \downarrow \\ \bigoplus_{x_{j+1} \in X_{j+1}^{(q-1)}} K_0^M(O_{X_{j+1}, x_{j+1}} \text{ on } x_{j+1}) & \xrightarrow{f_j^*} & \bigoplus_{x_j \in X_j^{(q-1)}} K_0^M(O_{X_j, x_j} \text{ on } x_j). \end{array}$$

This proves the deformation from $Z_{q, \text{rat}}^M(D^{\text{perf}}(X_j))$ to $Z_{q, \text{rat}}^M(D^{\text{perf}}(X_{j+1}))$ is unobstructed. In other words,

Theorem 4.10. $TZ_{\text{rat}}^q(X)$ is unobstructed.

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